PMM U.S.S.R.,Vol.51,No.6,pp.745-748,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

SEDIMENTATION OF A SUSPENSION OF SPHERICAL PARTICLES IN A CYLINDER*

O.B. GUS'KOV and A.V. ZOLOTOV

A method developed in /1-3/**(**See also: Struminskii V.V. et al. Laws of the mechanics of disperse media and two-phase systems in connection with the problems of improving the efficiency of technological processes. Pt.1, Preprint No.1, Moscow, Branch of Mechanics of Inhomogeneous Media, Academy of Sciences of the USSR, 1979.) is used in the creeping-flow approximation to study the problem of the sedimentation of a dilute monodisperse suspension of solid spherical particles inside an infinitely long cylindrical tube. The particles are statistically uniformly distributed in space. The following formula is obtained for the dimensionless velocity of the particles averaged over the ensemble, on the axis of the cylinder, in the limiting case when its walls are removed to infinity, to within first-order terms in the volume concentration c:

$$v_p = 1 - 4{,}675\ c \tag{0.1}$$

(the Stokes velocity of sedimentation of a single particle in an infinite fluid is taken as the characteristic velocity). It is shown that even when the expression obtained differs formally from the well-known result /4/ obtained for the mean sedimentation velocity of a limitless suspension

$$v_p = 1 - 6{,}55\ c \tag{0.2}$$

they nevertheless do not contradict each other.

Batchelor /4/ used the so-called method of renormalization in determining the mean velocity (0.2). This enables the well-known problem of the divergence of integrals in the averaging procedure to be avoided. However, the method still leaves open the question of what is the limiting physical situation realized experimentally, which corresponds to the concept of a limitless suspension adopted in /4/. The problem is directed related to the fact that the values of the suspension sedimentation velocity observed in the course of the experiments which are carried out, as a rule, in vertical cylindrical tubes, do not correspond to the result (0.2). Thus when all necessary conditions are satisfied (the sphericity of the particles, monodispersivity, homogeneity of the suspension and Stokes mode of sedimentation), the coefficient in front of the volume concentration of the particles varies, in the experimental relation of the velocity of sedimentation compatible with (0.2), from -4.65 to -4.80.

One of the aspects of this disparity between theory /4/ and experiment was revealed in /7, 8/ where an alternative approach was used to compute the mean velocity of the particles in the suspension, based on considering the motion of a collection of particles in the presence of physical boundaries, followed by averaging over the ensemble and removing the boundaries to infinity for constant volume concentration. When such an approach is used, problems of diverging integrals in the averaging procedure do not arise. Moreover, it is possible to show that the experimentally observed velocities of the particles should depend very much on the form of the vessel containing the suspension, even in the limit when its walls are infinitely distant. In particular, it was shown in /8/ that the relation (0.2) must correspond to the experimental values of the particle velocities in the case when the suspension settles on an infinitely distant plane wall. A different geometry of the experiment should give a different dependence of the velocity on the concentration.

Therefore the following formulas were obtained in /8/ in the problem of the settling of a suspension of rigid spherical particles inside a spherical vessel, when its radius tends to infinity, for a mean velocity of the particles and fluid v_p and v_f:

$$v_p = 1 - 3{,}55\ c,\quad v_f = 2\ c \tag{0.3}$$

It should be stressed that the result (0.3) is written in the laboratory system of coordinates attached to the fixed vessel, i.e. in the system where the experimental measurements are indirect. On passing to a system of coordinates moving relative to the vessel with mean volume velocity

$$v_v = cv_p + (1 - c)\ v_f = 3\ c$$

it is found that the difference $v_p - v_v = 1 - 6{,}55\ c$. The same equation is obtained /7/ in the case when the suspension settles on a flat surface. Thus in a system of coordinates where the total volume flux of the particles and liquid is zero, the relation (0.2) for the velocity

*Prikl.Matem.Mekhan.,51,6,968-972,1987

of the particles must hold irrespective of the geometry of the vessel.

An analogue of formula (0.3) is obtained below for the case when a suspension is settling in a vertical cylindrical tube.

1. Motion of a finite number of particles in a cylinder. Let us consider, in the Stokes approximation, a problem of quasistationary motion under the action of gravity, of N identical rigid spheres of radius a, in a viscous incompressible fluid bounded by a rigid cylindrical vessel of radius R^*. Let the velocities of the particles at some instant be $\mathbf{U}_i^*$ $(i = 1, 2, \ldots, N)$ and let their centres be situated at the points $\mathbf{r}_i^*$ inside the cylinder (an asterisk denotes a dimensional variable). We shall consider the case when the particles not acted upon by the moments of the forces from the direction of the liquid can rotate freely. The problem here is that of determining the velocities of the particles as a function of their mutual distribution in space.

We shall use the method described in /2/ to obtain an approximate solution of the problem. The method requires a preliminary determination of the hydrodynamic fields appearing, when there are no spheres, under the action of N point forces $\mathbf{F}_i^*$ and dipoles $\mathbf{D}_i^* = {}^1/_6 a^2 \mathbf{F}_i^*$ applied to the fluid at the points $\mathbf{r}_i^*$. The velocity and pressure fields must in this case satisfy the inhomogeneous system of equations /2/ with boundary conditions (in dimensionless form)

$$\Delta \mathbf{v} - \nabla p + \sum_{i=1}^{N} \{\mathbf{F}_i \delta(\mathbf{R}_i) - (\mathbf{D}_i \nabla) \nabla \delta(\mathbf{R}_i) + {}^1/_3 \mathbf{D}_i \Delta \delta(\mathbf{R}_i)\} = 0 \tag{1.1}$$

$$\nabla \cdot \mathbf{v} = {}^2/_3 \sum_{i=1}^{N} (\mathbf{D}_i \nabla) \delta(\mathbf{R}_i), \quad \mathbf{R}_i = \mathbf{r} - \mathbf{r}_i, \quad \mathbf{D}_i = {}^1/_6 \alpha^2 \mathbf{F}_i$$

$$\mathbf{v} \to 0, \quad z \to \pm\infty; \quad \mathbf{v} = 0, \quad \rho = R$$

$$\mathbf{r} = \frac{\mathbf{r}^*}{l}, \quad R = \frac{R^*}{l}, \quad p = \frac{p^* l}{\mu U}, \quad \mathbf{F}_i = \frac{\mathbf{F}_i^*}{\mu l U}, \quad \mathbf{D}_i = \frac{\mathbf{D}_i^*}{\mu l^3 U}$$

$$\mathbf{U}_i = \frac{\mathbf{U}_i^*}{U}, \quad \mathbf{v} = \frac{\mathbf{v}^*}{U}, \quad \alpha = \frac{a}{l}$$

Here $\delta(\mathbf{R}_i)$ is the Dirac delta function, ρ, φ, z are cylindrical coordinates with origin on the axis of the tube, μ is the dynamic viscosity, l is the characteristic distance between the particles and U is the Stokes velocity of settling of a single particle in an infinite fluid.

The general solution of system (1.1) can be written in the form

$$\mathbf{v} = \sum_{i=1}^{N} (\mathbf{u}_i + \mathbf{u}_i^W), \quad p = \sum_{i=1}^{N} (p_i + p_i^W) \tag{1.2}$$

where the functions $\mathbf{u}_i$ and p_i represent a special solution of system (1.1). The solution is the sum of the known hydrodynamic fields of point forces $\mathbf{F}_i$ and dipoles $\mathbf{D}_i$.

$$\mathbf{u}_i = \frac{1}{8\pi}\left[\frac{\mathbf{F}_i}{R_i} + \frac{(\mathbf{F}_i\mathbf{R}_i)\mathbf{R}_i}{R_i^3}\right] + \frac{1}{4\pi}\left[\frac{\mathbf{D}_i}{R_i^3} - \frac{3(\mathbf{D}_i\mathbf{R}_i)\mathbf{R}_i}{R_i^5}\right] \tag{1.3}$$

$$p_i = \frac{1}{4\pi}\frac{(\mathbf{F}_i\mathbf{R}_i)}{R_i^3}, \quad \mathbf{D}_i = {}^1/_6 \alpha^2 \mathbf{F}_i, \quad R_i = |\mathbf{R}_i|$$

Expressions for $\mathbf{u}_i^W$ and p_i^W for the given boundary conditions (1.1) were obtained in /9/.

Having found all the functions in (1.2), we obtain the velocities of the particles from the following relations /2/:

$$6\pi\alpha \mathbf{U}_i = \mathbf{F}_i + 6\pi\alpha [\mathbf{U}_0] + \pi\alpha^3 [\Delta \mathbf{U}_0] \tag{1.4}$$

$$\mathbf{U}_0 = \sum_{j \neq i}^{N} \mathbf{u}_j + \sum_{j=1}^{N} \mathbf{u}_j^W$$

where the vector $\mathbf{F}_i$ coincides with the resultant of the Archimedean and gravity forces acting on the particle. The square brackets in (1.4) indicate that the corresponding functions must be formally calculated at the centre of the i-th sphere.

It should be noted that the method described in /2/ gives an approximate solution of the problem. The perturbations caused by the particles in the stream are modelled, within the framework of this method, using the hydrodynamic fields of the point forces and dipoles only.

In order to obtain an exact solution we must take into account all higher-order multipoles. If, however, we take into account the results obtained in /4/, we do not need to know, in subsequent computations, the specific form of the functions describing the hydrodynamic multipole fields. Henceforth, we shall use the conventional notation for these functions.

Using the solution of /9/ for $\mathbf{u}_i^W$ and p_i^W, formulas (1.3) and the obvious relation $|\mathbf{F}_i| = F_{iz} = 6\pi\alpha$, we can obtain expressions for the velocities of the particles $\mathbf{U}_i$ and fluid $\mathbf{v}(\mathbf{r})$ in a specific form. In the special case when $\mathbf{r}_i = 0$ and $\mathbf{r} = 0$ respectively, the vertical components of the velocities will have the form

$$U_{iz} = 1 + \alpha\Big[\sum_{j\neq i}^{N} u_j^F + \sum_{j=1}^{N} u_j^{WF}\Big] + \tag{1.5}$$

$$\alpha^3\Big[2\sum_{j\neq i}^{N} u_j^D + \sum_{j=1}^{N} u_j^{WF} + \sum_{j=1}^{N} u_j^{WP}\Big] + \Delta_i + \Delta_i^W$$

$$v_z(\mathbf{r}=0) = \alpha\Big[\sum_{j\neq i}^{N} u_j^F + \sum_{j=1}^{N} u_j^{WF}\Big] + \alpha^3\Big[\sum_{j\neq i}^{N} u_j^D + \sum_{j=1}^{N} u_j^{WD}\Big] +$$

$$\Delta_f + \Delta_f^W$$

$$u_j^F = {}^3/_4\left(\frac{1}{r_j} + \frac{z_j^2}{r_j^3}\right), \quad u_j^D = {}^1/_4\left(\frac{1}{r_j^3} - \frac{3z_j^2}{r_j^5}\right)$$

$$u_j^{WF} = \frac{3}{2\pi R}\int_0^\infty Z_j^{(1)}\cos\left(\frac{\lambda z_j}{R}\right)d\lambda, \quad u_j^{WD} = \frac{1}{2\pi R^3}\int_0^\infty Z_j^{(2)}\cos\left(\frac{\lambda z_j}{R}\right)d\lambda$$

$$u_j^{WP} = \frac{1}{2\pi R^3}\int_0^\infty \lambda\Pi_j\cos\left(\frac{\lambda z_j}{R}\right)d\lambda, \quad Z_j^{(k)} = (\xi_j^{(k)}I_1 - \nu_j^{(k)}I_2)/Q$$

$$\Pi_j = (\nu_j^{(1)}I_1 - \xi_j^{(1)}I_0)/Q, \quad Q = I_1^2 - I_0I_2, \quad I_n = I_n(\lambda)$$

$$\xi_j^{(1)} = \beta_jS_1K_1 - \lambda S_0K_0, \quad \nu_j^{(1)} = \lambda S_0K_1 - \beta_jS_1K_0 - 2S_0K_0$$

$$\xi_j^{(2)} = \lambda^2S_0K_1, \quad \nu_j^{(2)} = -\lambda^2S_0K_0, \quad \beta_j = \lambda\rho_j/R$$

$$S_n = I_n(\beta_j), \quad K_n = K_n(\lambda)$$

where $I_n(x)$ and $K_n(x)$ are the modified n-th order Bessel function of the first and second kind, Δ_i and Δ_i^W are the contributions of the natural fields of the multipoles and the corresponding responses from the walls to the particle velocities, and Δ_f and Δ_f^W are the corresponding contributions to the velocity of the fluid.

2. Computing the mean sedimentation velocity. Let us consider the problem of the sedimentation of a group of rigid spheres in a fluid, along the vertical axis of the tube. We shall assume that the particles are statistically uniformly distributed throughout the fluid within a cylindrical layer of height H, i.e. within a volume whose boundaries are described by the equations $\rho = R$, $z = \pm H/2$. Using the linear approximation we shall find the mean velocity of the particles and the fluid in terms of the volume concentration, in the limit when first the height of the layer and then the volume of the cylinder both tend to infinity, with the value of c kept constant. Following /7, 8/, we shall determine the above velocities using the process of averaging over various particle configurations and calculating the mean values of the velocities at the centre of the layer

$$v_p = \lim\langle U_{iz} \mid \mathbf{r}_i = 0\rangle \tag{2.1}$$

$$v_f = \lim\langle v_z(\mathbf{r} = 0) \mid r_j > \alpha\rangle \tag{2.2}$$

Here the lim sign denotes the passage to the limit described above, $\langle\ldots|\,\mathbf{r}_i = 0\rangle$ denotes averaging over the possible configurations for which $\mathbf{r}_i = 0$. Similarly, $\langle\ldots|\,r_j > \alpha\rangle$ denotes averaging over the configurations for which not even a single particle overlaps the origin of coordinates. If we limit ourselves, in advance, to the linear approximation with respect to the volume concentrations and assume that the suspension is homogeneous, then the part of the weight function in the averaging process (2.1), (2.2) will be played by the numerical concentration equal to unity within the suitably chosen scales.

Substituting (1.5) into (2.1), we obtain

$$v_p = 1 + \lim\Big[\int_0^{2\pi} d\varphi_j \int_0^\infty \rho_j\,d\rho_j \int_{-\infty}^{\infty} dz_j\theta(R - \alpha - \rho_j)\times \tag{2.3}$$

$$\theta(H^2/4 - z_j^2)\,\theta(r_j - 2\alpha)\,\{\alpha(u_j^F + u_j^{WF}) +$$

$$\alpha^3(2u_j^D + u_j^{WD} + u_j^{WP})\} + \langle\Delta_i\,|\,\mathbf{r}_i = 0\rangle + \langle\Delta_i^W\,|\,\mathbf{r}_i = 0\rangle\Big]$$

where $\theta(x)$ is a step function equal to unity when $x \geqslant 0$, and to zero when $x < 0$. When computing the mean values of the quantities $\langle \Delta_i | \mathbf{r}_i = 0 \rangle$ and $\langle \Delta_i^W | \mathbf{r}_i = 0 \rangle$ in the linear approximation with respect to the concentration, we have to take into account in the functions Δ_i and Δ_i^W only the terms describing the pairwise interaction between the particles. Terms describing higher-order interactions will only contribute towards the coefficients of the higher-order concentration terms.

The quantity $\lim \langle \Delta_i | \mathbf{r}_i = 0 \rangle$ was calculated in the approximation of the pairwise interaction of the particles in /4/, and is equal to 1.55 c. We will find that within the same approximation the functions occurring in Δ_i^W are of the order of $(\rho_i/R)^n/R^m$ where $n + m \geqslant 4$. Therefore, after averaging Δ_i^W we obtain the value of zero in the limit as $R \to \infty$. Evaluating the remaining integrals and limits in (2.3) directly (when the effect of the walls is taken into account, the problem of divergence in the process of averaging (2.3) does not arise), and taking into account the obvious equality $4\pi\alpha^3 = 3c$, we can obtain the final expression for the mean velocity of the particles (0.1).

We find the mean velocity of the fluid at the centre of the layer (at $\mathbf{r} = 0$) in the same manner. Substituting (1.5) into (2.2) we obtain

$$v_f = \lim \Big[\int_0^{2\pi} d\varphi_j \int_0^{\infty} \rho_j \, d\rho_j \int_{-\infty}^{\infty} dz_j \theta (R - \alpha - \rho_j) \times \\ \theta (H^2/4 - z_j^2) \theta (r_j - \alpha) \{\alpha (u_j^F + u_j^{WF}) + \alpha^3 (u_j^D + u_j^{WD})\} + \\ \langle \Delta_f (\mathbf{r} = 0) | r_j > \alpha \rangle + \langle \Delta_f^W (\mathbf{r} = 0) | r_j > \alpha \rangle \Big] \tag{2.4}$$

The estimates obtained for Δ_i^W also apply to Δ_f^W. Therefore $\lim \langle \Delta_f^W (\mathbf{r} = 0) | r_j > \alpha \rangle = 0$. The function Δ_f also makes a zero contribution to the velocity of the liquid in the linear approximation with respect to concentration. This assertion follows directly from the fact that when a single particle moves in an infinite fluid, the function describing the velocity field does not contain terms of the order of r_j^{-n} when $n \geqslant 4$ (the two-particle interactions make a contribution to the velocity of the liquid only in the coefficients in front of c^k when $k \geqslant 2$). Carrying out direct calculations in (2.4) we obtain

$$v_f = {}^7/_8 \; c \tag{2.5}$$

Thus formulas (0.1) and (2.5) hold for the velocities of the particles and liquid on the cylinder axis, in the limiting case when the walls of the cylinder are removed to infinity.

Formally, result (0.1) agrees with experimental data /5, 6/. This, however, does not fully clarify the problem of interpreting such experiments. Usually, in experiments dealing with the settling of suspensions in cylindrical tubes one follows the motion of the boundary separating the suspension from the clear liquid, and the stability of this boundary represents the characteristic feature of this phenomenon. Therefore, the next stage in explaining similar experiments might consist of a theoretical study of the problems of the evolution and stabilization of the boundary of separation.

Expressions (0.1) and (2.5) for the velocities of the particles and the fluid are obtained in a system of coordinates attached to the fixed walls of the cylinder. Let us change to a system of coordinates moving relative to the cylinder at a mean volume rate of $v_v = cv_p + (1 - c) v_f = {}^{15}/_8 \; c$.

The expression for the velocity of the particles in this system, as might be expected, is identical with result (0.2).

REFERENCES

1. STRUMINSKII V.V., GUS'KOV O.B. and KUL'BITSKII YU.N., Hydrodynamics of disperse and gas-liquid flows. Dokl. Akad. Nauk SSSR, 278, 1, 1984.
2. GUS'KOV O.B. and STRUMINSKII V.V., Dynamics of disperse flows in the presence of the boundaries. Dokl. Akad. Nauk SSSR, 285, 4, 1985.
3. GUS'KOV O.B., Theory of hydrodynamic interaction between spherical fluid drops in viscous flow. Modern Problems of Thermal Physics. Novosibirsk, In-t teplofiziki, Siberian Branch of the Academy of Sciences of the SSSR, 1984.
4. BATCHELOR G.K., Sedimentation in a dilute dispersion of spheres, J. Fluid Mech. 52, 2, 1972.
5. RICHARDSON J.F. and ZAKI W.N., The sedimentation of a suspension of uniform spheres under conditions of viscous flow, Chem. Eng. Sci. 3, 2, 1954.
6. CHONG Y.S., RATCOWSKY D.A. and EPSTEIN N., Effect of particle shape on hindered settling in creeping flow, Powder Technol. 23, 1, 1979.
7. BEENAKKER C.W.J. and MASUR P., On the Smoluchowskii paradox in a sedimenting suspension, Phys. Fluids. 28, 3, 1985.
8. BEENAKKER C.W.J. and MASUR P., Is sedimentation container-shape dependent? Phys. Fluids, 28, 11, 1985.
9. GUS'KOV O.B., On the interaction between the particles in a viscous fluid in the presence of boundaries. Hydrodynamic Problems of Technological Processes. Moscow, Nauka, 1988.

Translated by L.K.